Towards Synthetic Descriptive Set Theory: An instantiation with represented spaces

Arno Pauly [1]   Matthew de Brecht [2]

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[1] University of Cambridge, United Kingdom
[2] National Institute of Information and Communications Technology, Japan

Outline

Basics of descriptive set theory

- Let $\Sigma_1^0(\mathbf{X}) = \mathcal{O}(\mathbf{X})$.
- Let $\Pi_\alpha^0(\mathbf{X}) = \{X \setminus U \mid U \in \Sigma_\alpha^0(\mathbf{X})\}$.
- Let $\Sigma_{\alpha+1}^0(\mathbf{X}) = \{\left(\bigcup_{n\in\mathbb{N}} A_n\right) \mid A_n \in \Pi_\alpha^0(\mathbf{X})\}$.
- Let $\Delta_\alpha^0(\mathbf{X}) = \Sigma_\alpha^0(\mathbf{X}) \cap \Pi_\alpha^0(\mathbf{X})$
- A function f is called $\mathfrak{B}$-measurable, if $f^{-1}(U) \in \mathfrak{B}$ for any $U \in \mathcal{O}(\mathbf{Y})$.

Banach Hausdorff Lebesgue theorem

Theorem (BANACH, LEBESGUE, HAUSDORFF)

The Σ^0_{n+1}-measurable functions between separable metric spaces are exactly the pointwise limits of Σ^0_n-measurable functions[3]*.*

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[3] Restrictions apply

Some fundamental results II

Definition

$f : \mathbf{X} \to \mathbf{Y}$ is piecewise continuous, if there is a closed cover $(A_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$ such that any $f_{|A_n}$ is continuous.

Theorem (Jayne & Rogers)

Let $\mathbf{X}$, $\mathbf{Y}$ be Polish spaces. A function $f : \mathbf{X} \to \mathbf{Y}$ is Δ^0_2-measurable, iff it is piecewise continuous.

Represented spaces and computability

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ a surjective partial function.

Definition

$f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a realizer of $F : \mathbf{X} \to \mathbf{Y}$, iff $F(\delta_X(p)) = \delta_Y(f(p))$ for all $p \in \delta_X^{-1}(\mathrm{dom}(F))$.

$$
\begin{array}{ccc}
\mathbb{N}^{\mathbb{N}} & \xrightarrow{f} & \mathbb{N}^{\mathbb{N}} \\
\downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} \\
\mathbf{X} & \xrightarrow{F} & \mathbf{Y}
\end{array}
$$

Definition

$F : \mathbf{X} \to \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

Endofunctor

An endofunctor d is an operation on a category, mapping objects to objects, identities to identities and morphisms to morphisms that respects composition.

We shall pretend that in a cartesian closed category with exponentials $\mathcal{E}$, for any two fixed objects A, B an endofunctor d induces a map $d : \mathcal{E}(A, B) \to \mathcal{E}(dA, aB)$.

The jump of a represented space

Consider $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined via $\lim(p)(i) = \lim_{j\to\infty} p(\langle i,j\rangle)$.

Definition (ZIEGLER)

Given a represented space $\mathbf{X} = (X, \delta_{\mathbf{X}})$, introduce $\mathbf{X}' = (X, \delta_{\mathbf{X}} \circ \lim)$.

Proposition (ZIEGLER)

The lifting map $\mathrm{id} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \to \mathcal{C}(\mathbf{X}', \mathbf{Y}')$ *is well-defined and computable.*

More on the jump

Theorem (Brattka)

The following are equivalent for $f : \mathbf{X} \to \mathbf{Y}$*, with* $\mathbf{X}$*,* $\mathbf{Y}$ *CMS:*

1. $f \leq_W \lim$ *relative to some oracle*
2. f *is* Σ^0_2*-measurable*
3. $f : \mathbf{X} \to \mathbf{Y}'$ *is continuous*

Remark: 2. is a *backward*-notion, while 3. is a *forward* notion.

Realizer vs topological continuity

Proposition

The map $f \mapsto f^{-1} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \to \mathcal{C}(\mathcal{O}(\mathbf{Y}), \mathcal{O}(\mathbf{X}))$ is computable for all represented spaces $\mathbf{X}$, $\mathbf{Y}$.

Remark: Continuity for represented spaces is a forwards notion, topological continuity a backwards notion.

Admissibility

Definition (SCHRÖDER)

Call $\mathbf{X}$ (computably) admissible, if the canonic map $\kappa : \mathbf{X} \to \mathcal{C}(\mathcal{O}(\mathbf{X}), \mathbb{S})$ is (computably) continuously invertible. κ maps x to $U \mapsto U(x)$.

Theorem (SCHRÖDER)

$\mathbf{Y}$ is (computably) admissible, iff for any $\mathbf{X}$ the map $f \mapsto f^{-1} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \to \mathcal{C}(\mathcal{O}(\mathbf{Y}), \mathcal{O}(\mathbf{X}))$ is (computably) continuously invertible.

Remark: So admissibility makes forwards and backwards notions coincide.

Computable endofunctors and basic notions

Definition
An endofunctor d on the category of represented spaces is called *computable*, iff for any represented spaces $\mathbf{X}, \mathbf{Y}$ the induced map $d : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \to \mathcal{C}(d\mathbf{X}, d\mathbf{Y})$ is computable. (Tacit assumption: d does not change the underlying sets.)

Definition
Call $f : \mathbf{X} \to \mathbf{Y}$ d-continuous, iff $f : \mathbf{X} \to d\mathbf{Y}$ is continuous.

Definition
Call $U \subseteq \mathbf{X}$ d-open, iff $\chi_U : \mathbf{X} \to d\mathbb{S}$ is continuous. The space of d-opens is $\mathcal{O}^d(\mathbf{X})$.

Definition
Call $f : \mathbf{X} \to \mathbf{Y}$ d-measurable, iff $f^{-1} : \mathcal{O}(\mathbf{Y}) \to \mathcal{O}^d(\mathbf{X})$ is continuous.

A first observation

Proposition

Any d-continuous function is d-measurable.

Definition

Call $\mathbf{Y}$ d-admissible, if the canonic map $\kappa^d : d\mathbf{Y} \to \mathcal{C}(\mathcal{C}(\mathbf{Y}, \mathbb{S}), d\mathbb{S})$ is computably invertible.

Theorem

If $\mathbf{Y}$ is d-admissible, then for functions $f : \mathbf{X} \to \mathbf{Y}$ d-continuity and d-measurability coincide.

Some structural properties

Theorem

Let d satisfy ($d(\mathbf{X} \times \mathbf{X}) \cong d\mathbf{X} \times d\mathbf{X}$) ($d\mathcal{C}(\mathbb{N}, \mathbf{X}) = \mathcal{C}(\mathbb{N}, d\mathbf{X})$) for all represented spaces $\mathbf{X}$,$\mathbf{Y}$. We may conclude:

1. $(f, U) \mapsto f^{-1}(U) : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \times \mathcal{O}^d(\mathbf{Y}) \to \mathcal{O}^d(\mathbf{X})$ *is well-defined and computable.*
2. $\cap, \cup : \mathcal{O}^d(\mathbf{X}) \times \mathcal{O}^d(\mathbf{X}) \to \mathcal{O}^d(\mathbf{X})$ *are well-defined and computable.*
3. *Any countably based admissible space* $\mathbf{X}$ *is d-admissible.*
4. $\bigcup : \mathcal{C}(\mathbb{N}, \mathcal{O}^d(\mathbf{X})) \to \mathcal{O}^d(\mathbf{X})$ *is well-defined and computable.*

The jump operator

Proposition

$'$ is a computable endofunctor satisfying $\mathcal{C}(\mathbb{N}, \mathbf{X})' \cong \mathcal{C}(\mathbb{N}, \mathbf{X}')$.

Proposition

The map $(U_i)_{i\in\mathbb{N}} \mapsto \bigcup_{i\in\mathbb{N}}(X \setminus U_i) : \mathcal{C}(\mathbb{N}, \mathcal{O}(\mathbf{X})) \to \mathcal{O}'(\mathbf{X})$ *is computable. If* $\mathbf{X}$ *is a computable metric space, then it is even computably invertible.*

Corollary

For a computable metric space $\mathbf{X}$, $\Sigma^0_2(\mathbf{X}) = \mathcal{O}'(\mathbf{X})$.

Banach Lebesgue Hausdorff Theorem

Corollary (Banach Lebesgue Hausdorff Theorem)

Any countably-based admissible space $\mathbf{X}$ *is* $'$*-admissible, i.e.* $^{-1} : \mathcal{C}(\mathbf{X}, \mathbf{Y}') \to \mathcal{C}(\mathcal{O}(\mathbf{Y}), \mathcal{O}'(\mathbf{X}))$ *is computable and computably invertible.*

Changing the sets

Consider the computable endofunctor $\mathcal{K}$ mapping a space to the space of its compact subsets.

Observation

The $\mathcal{K}$-continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ are just the upper hemicontinuous multivalued functions from $\mathbf{X}$ to $\mathbf{Y}$.

The finite mindchange endofunctor

Definition

Define $\nabla :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ via $\nabla(w0p) = p - 1$ iff p contains no 0.
Define an operator $^\nabla$ via $(X, \delta_X)^\nabla = (X, \delta_X \circ \nabla)$.

Observation

∇ is a computable endofunctor preserving binary products.

Proposition

Let $\mathbf{X}, \mathbf{Y}$ be computable metric spaces. Then $f : \mathbf{X} \to \mathbf{Y}$ is piecewise continuous iff $f : \mathbf{X} \to \mathbf{Y}^\nabla$ is continuous.

Proposition

$\mathcal{O}'(\mathbf{X}) \cap \mathcal{A}'(\mathbf{X}) = \mathcal{O}^\nabla(\mathbf{X})$

Back to the abstract picture

Definition

We call a space $\mathbf{X}$ d-Hausdorff, iff $x \mapsto \{x\} : \mathbf{X} \to \mathcal{A}^d(\mathbf{X})$ is computable.

Observation

Being ∇-Hausdorff corresponds to the T_D separation axiom.

The effective Jayne Rogers theorem

Theorem

If $\mathbf{Y}$ *has a total representation* $\delta_{\mathbf{Y}} : \{0,1\}^{\mathbb{N}} \to \mathbf{Y}$ *and is* ∇*-Hausdorff, then it is* ∇*-admissible.*

Corollary

For computable metric spaces, $f : \mathbf{X} \to \mathbf{Y}$ *is (uniformly)* Δ^0_2*-measurable, iff it is piecewise continuous.*

The proof

We need to show that $(x, f^{-1}) \mapsto f(x) : \mathbf{X} \times \mathcal{C}(\mathcal{O}(\mathbf{Y}), \Delta^0_2(\mathbf{X})) \to \mathbf{Y}^\nabla$ is computable. To do this, show that $(x, f^{-1}) \mapsto f(x) : \mathbf{X} \times \mathcal{C}(\mathcal{O}(\mathbf{Y}), \Delta^0_2(\mathbf{X})) \to \mathbf{Y}$ is non-deterministically computable with advice $\{0, 1\}^{\mathbb{N}} \times \mathbb{N}$ and use:

Theorem (Brattka, de Brecht & P.)

If $f : \mathbf{X} \to \mathbf{Y}$ is single-valued and non-deterministically computable with advice $\{0, 1\}^{\mathbb{N}} \times \mathbb{N}$, then it is computable with advice $\mathbb{N}$.

Proposition (Brattka, de Brecht & P.)

A function is non-deterministically computable with advice $\mathbb{N}$, iff it is computable with finitely many mindchanges.

The algorithm

1. Guess $n \in \mathbb{N}$ and $p \in \{0, 1\}^{\mathbb{N}}$ encoding some $y \in \mathbf{Y}$.
2. Compute $\mathbf{Y} \setminus \{y\} \in \mathcal{O}(\mathbf{Y})$.
3. Compute $f^{-1}(\mathbf{Y} \setminus \{y\}) = \bigcap_{i \in \mathbb{N}} O_i$ (more generally $= A \in \mathcal{A}'(\mathbf{X})$).
4. Test $x \in O_i$ for all $i \leq n$ (evaluate the first n approximations of $A(x)$), and reject if all answers are positive.
5. Output y.

Counterexamples

Example

There is a function $f :\subseteq \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that for any computably open set $U \subseteq \{0,1\}^{\mathbb{N}}$ the set $f^{-1}(U)$ is effectively Δ^0_2, yet f is not even non-uniformly computable.

Example

There are countably based quasi-Polish spaces that are not ∇-admissible.

More synthetic?

Can properties of specific endofunctors on represented spaces such as $'$ be explained by generic characterizations, e.g. $'$ being the minimal computable endofunctor above id preserving countable products?

Understanding represented spaces

What represented spaces have total Cantor space representations? What other (new) properties of spaces are relevant for this approach to descriptive set theory?

Understanding the projective hierarchy

How does the endofunctor for the projective hierarchy look like?
To what extent can Suslin's theorem that $\Delta^1_1 = \bigcup_\alpha \Sigma^0_\alpha$ be generalized?